# Bell's inequality for the Mach-Zehnder interferometer

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We show that no local, hidden variable model can be given for two-channel states exhibiting both a sufficiently high interference visibility *and* a sufficient degree of anticorrelation in a Mach-Zehnder interferometer.

## I. INTRODUCTION

If a single photon impinges on a Mach-Zehnder interferometer, the probability of detecting the photon in a certain output channel depends on the difference between the phase delays imposed by the two interfering channels. The photon behaves as if it goes through both interfering channels. On the other hand, if detectors are inserted into each interfering channel, the photon is only detected in one of the channels. In the delayed-choice experiment, the detectors may even be inserted in the last instant. Then the experimenter's choice whether to insert these detectors or not determines whether the photon shall behave "as if" it goes through one or both interfering channels.

It might seem that the detectors impart some sort of "nonlocal collapse" to the photon state. Otherwise, one may imagine that the photon itself goes through one channel only accompanied by an "empty wave" in the other channel. Such models have been developed by de Broglie [1] and later by Bohm [2]. They are known to be nonlocal.

A common feature of both the "collapse" picture and the "empty wave" picture is that they refer to systems which appear to be "delocalized" (yielding interference) in a certain experiment and "localized" (yielding anticorrelation) in another. It may seem that such behavior is in some way nonlocal.

Although classical optical fields may display interference effects in the same way as single photons, they cannot simultaneously be anticorrelated. One may also imagine stochastic classical fields which yield anticorrelation between two channels, but such two-channel systems may not give rise to interference effects when superposed.

One might suspect that two-channel quantum states which yield *both* interference *and* anticorrelation in a Mach-Zehnder interferometer violate local realism. In this paper we shall see that this is indeed the case, and that for such systems no local hidden variable model [3] can be constructed.

## II. THE MACH-ZEHNDER INTERFEROMETER

Before deriving the Bell inequality, we first need to specify the characteristic quantities that we observe in a Mach-Zehnder interferometer (see Fig. 1).

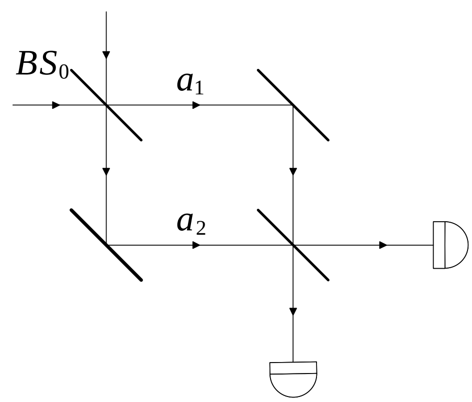


FIG. 1. Mach-Zehnder interferometer where the degree of first and second order coherence between the *a*-channels may be observed directly.

It is well known that the interference visibility obtainable in a first order interferometer is represented by the degree of first order coherence [4]

$$g^{(1)} = \frac{\langle \hat{a}_1^\dagger \hat{a}_2 \rangle}{\sqrt{\langle \hat{a}_1^\dagger \hat{a}_1 \rangle \langle \hat{a}_2^\dagger \hat{a}_2 \rangle}}. \tag{1}$$

Here $\hat{a}_k$ and $\hat{a}_k^\dagger$ are annihilation and creation operators for the interfering channel $a_k$ $(k = 1, 2)$ of the interferometer. For simplicity we employ a single mode treatment.

If, on the other hand, we insert detectors into each channel of the interferometer, we may observe the coincidence rate or the degree of second order coherence between the same two channels [4],

$$g^{(2)} = \frac{\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 \rangle}{\langle \hat{a}_1^\dagger \hat{a}_1 \rangle \langle \hat{a}_2^\dagger \hat{a}_2 \rangle}. \tag{2}$$

The fact that classical field theories do not allow both interference and anticorrelation for the same system is

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illustrated by the fact that in these theories the inequality [5]

$$| g^{(1)} |^2 \leq g^{(2)} \tag{3}$$

must be fulfilled.

## III. BELL'S INEQUALITY

We first analyze the experiment shown in Fig. 2. We shall see that the observables here are closely related to the Mach-Zehnder interferometer (Fig. 1).

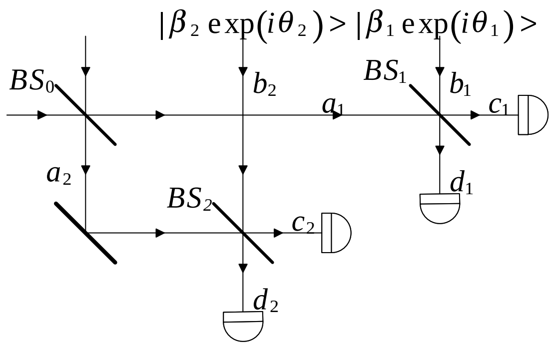


FIG. 2. An experiment where the two $a$-channels are mixed with with coherent, homodyne local oscillators. In this experiment, local realism imposes constraints on the degree of first and second order coherence between the $a$-channels.

Grangier *et al.* [6] were the first to propose the use of local oscillators in Bell experiments. The Bell experiment that we will use here (Fig. 2) was first proposed by Oliver and Stroud [7]. They also showed that single photon states violate local realism in this interferometer. Later Tan, Holland and Walls (THW) [8] performed a thorough derivation of the conditions for local realism for any state in this interferometer. The behavior of single photon states in this interferometer was treated extensively by Tan, Walls and Collett [9].

We shall generalize the work of THW. Again, we restrict the attention to single-mode systems. Let $\hat{\mu}_k$ and $\hat{\mu}_k^\dagger$ be annihilation and creation operators for the channel $\mu_k$ ($\mu = a, b, c, ...,\ k = 1, 2$). The $a$-channels are output channels from beamsplitter $BS_0$, in analogy with the Mach-Zehnder interferometer (Fig. 1). Each channel $a_k$ is mixed with a local oscillator channel $b_k$ on a semireflecting beamsplitter $BS_k$. The local oscillator is represented as a coherent state $| \beta_k \exp(i\theta_k)\rangle$ with (real) amplitude $\beta_k$ and (real) phase $\theta_k$.

The connection between the input and output on beamsplitter $BS_k$ is given by the transformation

$$\begin{pmatrix} \hat{c}_k \\ \hat{d}_k \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_k \\ \hat{b}_k \end{pmatrix}. \tag{4}$$

We find that the total photon number is preserved,

$$\hat{S}_k = \hat{c}_k^\dagger \hat{c}_k + \hat{d}_k^\dagger \hat{d}_k = \hat{a}_k^\dagger \hat{a}_k + \hat{b}_k^\dagger \hat{b}_k. \tag{5a}$$

Also, we may rewrite the difference between photon numbers at the two beamsplitter output channels in terms of input operators,

$$\hat{D}_k = \hat{c}_k^\dagger \hat{c}_k - \hat{d}_k^\dagger \hat{d}_k = i\left(\hat{a}_k^\dagger \hat{b}_k - \hat{a}_k \hat{b}_k^\dagger\right). \tag{5b}$$

We will now study the quantity

$$E(\theta_1, \theta_2) = \frac{\langle \hat{D}_1 \hat{D}_2 \rangle}{\langle \hat{S}_1 \hat{S}_2 \rangle}, \tag{6}$$

which may be termed the "modulation depth" of the correlation between the two interferometers. Note that it's modulus is restricted from above to unity.

Using the operator definitions (5) we find

$$\begin{aligned} E(\theta_1, \theta_2) = & \frac{\beta_1 \beta_2}{\langle \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 \rangle + \langle \hat{a}_1^\dagger \hat{a}_1 \rangle \beta_2^2 + \langle \hat{a}_2^\dagger \hat{a}_2 \rangle \beta_1^2 + \beta_1^2 \beta_2^2} \\ & \times \Big[ \langle \hat{a}_1^\dagger \hat{a}_2 \rangle \exp[i(\theta_1 - \theta_2)] + \langle \hat{a}_1 \hat{a}_2^\dagger \rangle \exp[-i(\theta_1 - \theta_2)] \\ & - \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle \exp[i(\theta_1 + \theta_2] - \langle \hat{a}_1 \hat{a}_2 \rangle \exp[-i(\theta_1 + \theta_2)] \Big]. \end{aligned} \tag{7}$$

The local oscillator amplitudes are of course independent parameters, and may be chosen freely. We now want to choose them so that the modulation depth is maximized. It can be shown that this is achieved by the choice

$$\beta_1 \beta_2 = \sqrt{\langle \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 \rangle}, \quad \frac{\beta_1}{\beta_2} = \sqrt{\frac{\langle \hat{a}_1^\dagger \hat{a}_1 \rangle}{\langle \hat{a}_2^\dagger \hat{a}_2 \rangle}}. \tag{8}$$

Inserting this into Eq. (7) we may write

$$\begin{aligned} E(\theta_1, \theta_2) = & C_1 \cos(\theta_1 - \theta_2 + \arg\langle \hat{a}_1^\dagger \hat{a}_2 \rangle) \\ & + C_2 \cos(\theta_1 + \theta_2 + \arg\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle). \end{aligned} \tag{9}$$

We are particularly interested in the coefficient $C_1$, since it may be expressed in terms of well known coherence functions between the two $a$-channels,

$$C_1 = \frac{| g^{(1)} |}{1 + \sqrt{g^{(2)}}}. \tag{10}$$

If we consider the quantity

$$B = E(\theta_1, \theta_2) + E(\theta_1, \theta_2') + E(\theta_1', \theta_2) - E(\theta_1', \theta_2'), \tag{11}$$

local realism imposes the restriction [10]

$$| B | \leq 2. \tag{12}$$

THW [8] showed that this is satisfied whenever

$$C_1^2 + C_2^2 \leq 1/2. \tag{13}$$

Therefore a minimal (necessary, but not sufficient) requirement for local realism is that

$$C_1 \leq 1/\sqrt{2}, \tag{14}$$

or

$$\frac{\mid g^{(1)} \mid}{1+\sqrt{g^{(2)}}} \leq 1/\sqrt{2}. \tag{15}$$

This may be considered as a rewriting of the Bell inequality (12). We see that violation of this inequality implies that inequality (3) is also violated. However, inequality (3) must be strongly violated in order to imply violation of inequality (15). In other words, if the state violates local realism, it also violates classical field theory, but the converse is not necessarily true.

## IV. BELL'S INEQUALITY FOR THE MACH-ZEHNDER INTERFEROMETER

We of course note that the parameters involved in Bell's inequality (15) are exactly the same that we observe in the Mach-Zehnder interferometer, namely the degree of first and second order coherence. In other words, this inequality involves the interference visibility and the coincidence rate for a Mach-Zehnder interferometer. This of course also means that we can *test* this inequality in a Mach-Zehnder interferometer.

It can be noted that in order to observe violation of inequality (15), a minimal requirement is that

$$\begin{aligned} &g^{(1)} > 1/\sqrt{2} \approx 0.71, \\ &g^{(2)} < (\sqrt{2}-1)^2 \approx 0.17. \end{aligned} \tag{16}$$

We see that the state must be both sufficiently anticorrelated and it must yield a sufficiently high interference visibility. We see that this corresponds with our suspicion that the combination of these two features in some way yields violation of local realism.

The most extreme example in this respect is of course the split single photon state

$$\mid \psi \rangle_a = \frac{1}{\sqrt{2}} \left( \mid 1 \rangle_{a_1} \mid 0 \rangle_{a_2} + i \mid 0 \rangle_{a_1} \mid 1 \rangle_{a_2} \right), \tag{17}$$

which yields $g^{(1)} = 1$ and $g^{(2)} = 0$. This state has been shown to violate local realism [7–9] in the same experiment as we consider here. But in this paper we have in addition seen that a whole class of states violates local realism, including also mixed states. These states possess certain common features, namely those of a high interference visibility in combination with a strong anticorrelation.

Grangier, Roger and Aspect [11] have performed an experiment measuring the visibility and the coincidence rate in the Mach-Zehnder interferometer. They observed a visibility of 0.98 and a coincidence rate of 0.18. Although this is sufficient to demonstrate violation of inequality (3), the coincidence rate was slightly too high to demonstrate violation of inequality (15). However, such a demonstration should be well within technological reach today.

Note that a direct contradiction with local realism is not achieved in a Mach-Zehnder interferometer. If inequality (3) is violated, the experiments do show that classical field theories break down. However, even if inequality (15) is violated, the results can be explained by a *local* but *contextual* hidden variable model. We may, e.g., explain the interference visibility in terms of a classical wave model and the anticorrelation in terms of a classical particle model.

Still, it is interesting to see that merely by observing the interference visibility and the coincidence rate on an unknown state, we may gain sufficient information to predict that this state will violate local realism in the THW-experiment. Thus, any single-mode state, pure or mixed, which displays both a sufficiently high interference visibility and a sufficient degree of anticorrelation violates local realism.

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